HL Paper 3

A relation S is defined on $\mathbb{R}$ by aSb if and only if $ab > 0$.

A relation R is defined on a non-empty set A. R is symmetric and transitive but not reflexive.

a. Show that S is [4]

(i) not reflexive;

(ii) symmetric;

(iii) transitive.

b. Explain why there exists an element $a \in A$ that is not related to itself. [1]

c. Hence prove that there is at least one element of A that is not related to any other element of A. [6]

Let $f : G \to H$ be a homomorphism between groups $\{G,\ *\}$ and $\{H,\ \circ\}$ with identities e_G and e_H respectively.

a. Prove that $f(e_G) = e_H$. [2]

b. Prove that $\text{Ker}(f)$ is a subgroup of $\{G,\ *\}$. [6]

A, B and C are three subsets of a universal set.

Consider the sets $P = \{1,\ 2,\ 3\},\ Q = \{2,\ 3,\ 4\}$ and $R = \{1,\ 3,\ 5\}$.

a.i. Represent the following set on a Venn diagram, [1]

$A \Delta B$, the symmetric difference of the sets A and B;

a.ii. Represent the following set on a Venn diagram, [1]

$A \cap (B \cup C)$.

b.i. For sets P, Q and R, verify that $P \cup (Q \Delta R) \neq (P \cup Q) \Delta (P \cup R)$. [4]

b.ii. In the context of the distributive law, describe what the result in part (b)(i) illustrates. [2]

The function $f:\mathbb{Z}\to\mathbb{Z}$ is defined by $f(n)=n+(-1)^n$.

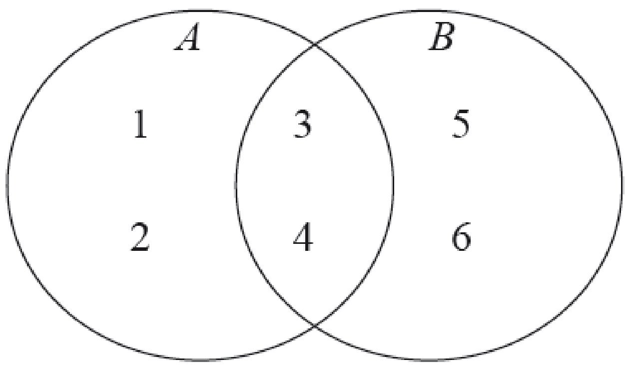
a. Prove that $f\circ f$ is the identity function. [6]

b.i. Show that f is injective. [2]

b.ii. Show that f is surjective. [1]

Let $\{G,\ \circ\}$ be the group of all permutations of $1,\ 2,\ 3,\ 4,\ 5,\ 6$ under the operation of composition of permutations.

Consider the following Venn diagram, where $A=\{1,\ 2,\ 3,\ 4\},\ B=\{3,\ 4,\ 5,\ 6\}$.



a. (i) Write the permutation $\alpha=\begin{pmatrix}1&2&3&4&5&6\\3&4&6&2&1&5\end{pmatrix}$ as a composition of disjoint cycles. [3]

(ii) State the order of α.

b. (i) Write the permutation $\beta=\begin{pmatrix}1&2&3&4&5&6\\6&4&3&5&1&2\end{pmatrix}$ as a composition of disjoint cycles. [2]

(ii) State the order of β.

c. Write the permutation $\alpha\circ\beta$ as a composition of disjoint cycles. [2]

d. Write the permutation $\beta\circ\alpha$ as a composition of disjoint cycles. [2]

e. State the order of $\{G,\ \circ\}$. [2]

f. Find the number of permutations in $\{G,\ \circ\}$ which will result in A, B and $A\cap B$ remaining unchanged. [2]

The binary operations $\odot$ and $*$ are defined on $\mathbb{R}^+$ by

$$a\odot b=\sqrt{ab}\text{ and }a*b=a^2b^2.$$

Determine whether or not

a. $\odot$ is commutative; [2]

b. $*$ is associative; [4]

c. $*$ is distributive over $\odot$; [4]

d. $\odot$ has an identity element. [3]

Let $\{G,\ *\}$ be a finite group that contains an element a (that is not the identity element) and $H = \{a^n | n \in \mathbb{Z}^+\}$, where

$a^2 = a * a,\ a^3 = a * a * a$ etc.

Show that $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$.

The set A contains all positive integers less than 20 that are congruent to 3 modulo 4.

The set B contains all the prime numbers less than 20.

The set C is defined as $C = \{7,\ 9,\ 13,\ 19\}$.

a.i. Write down all the elements of A and all the elements of B. [2]

a.ii. Determine the symmetric difference, $A\Delta B$, of the sets A and B. [2]

b.i. Write down all the elements of $A \cap B,\ A \cap C$ and $B \cup C$. [3]

b.ii. Hence by considering $A \cap (B \cup C)$, verify that in this case the operation $\cap$ is distributive over the operation $\cup$. [3]

The relation *R* is defined on $\mathbb{R} \times \mathbb{R}$ such that $(x_1,\ y_1)R(x_2,\ y_2)$ if and only if $x_1y_1 = x_2y_2$.

a. Show that *R* is an equivalence relation. [5]

b. Determine the equivalence class of *R* containing the element $(1,\ 2)$ and illustrate this graphically. [4]

The group $\{G,\ \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where $\times_7$ denotes multiplication modulo 7.

a. (i) Write down the Cayley table for $\{G,\ \times_7\}$. [10]

(ii) Determine whether or not $\{G,\ \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H.

b. The group $\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and $\circ$ denotes composition of permutations. [6]

(i) Show that $\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not $\{G, \times_7\}$ and $\{K, \circ\}$ are isomorphic.

The set of all permutations of the elements $1, 2, \ldots 10$ is denoted by H and the binary operation $\circ$ represents the composition of permutations.

The permutation $p = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$ generates the subgroup $\{G, \circ\}$ of the group $\{H, \circ\}$.

a. Find the order of $\{G, \circ\}$. [2]

b. State the identity element in $\{G, \circ\}$. [1]

c. Find [4]

(i) $p \circ p$;

(ii) the inverse of $p \circ p$.

d. (i) Find the maximum possible order of an element in $\{H, \circ\}$. [3]

(ii) Give an example of an element with this order.

The relation R is defined on the set $\mathbb{N}$ such that for a, $b \in \mathbb{N}$, aRb if and only if $a^3 \equiv b^3 (\text{ mod } 7)$.

a. Show that R is an equivalence relation. [6]

b. Find the equivalence class containing 0. [2]

c. Denote the equivalence class containing n by C_n. [3]

List the first six elements of C_1.

d. Denote the equivalence class containing n by C_n. [3]

Prove that $C_n = C_{n+7}$ for all $n \in \mathbb{N}$.

The function f is defined by $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ where $f(x, y) = \left(\sqrt{xy}, \frac{x}{y}\right)$

a. Prove that f is an injection. [5]

b. (i) Prove that f is a surjection. [8]

(ii) Hence, or otherwise, write down the inverse function f^{-1}.

The relation $R =$ is defined on $\mathbb{Z}^+$ such that aRb if and only if $b^n - a^n \equiv 0(\bmod p)$ where $n,\ p$ are fixed positive integers greater than 1.

a. Show that R is an equivalence relation. [7]

b. Given that $n = 2$ and $p = 7$, determine the first four members of each of the four equivalence classes of R. [5]

Let c be a positive, real constant. Let G be the set $\{x \in \mathbb{R} | -c < x < c\}$. The binary operation $*$ is defined on the set G by $x * y = \frac{x+y}{1+\frac{xy}{c^2}}$.

a. Simplify $\frac{c}{2} * \frac{3c}{4}$. [2]

b. State the identity element for G under $*$. [1]

c. For $x \in G$ find an expression for x^{-1} (the inverse of x under $*$). [1]

d. Show that the binary operation $*$ is commutative on G . [2]

e. Show that the binary operation $*$ is associative on G . [4]

f. (i) If $x,\ y \in G$ explain why $(c - x)(c - y) > 0$. [2]

(ii) Hence show that $x + y < c + \frac{xy}{c}$.

g. Show that G is closed under $*$. [2]

h. Explain why $\{G, *\}$ is an Abelian group. [2]

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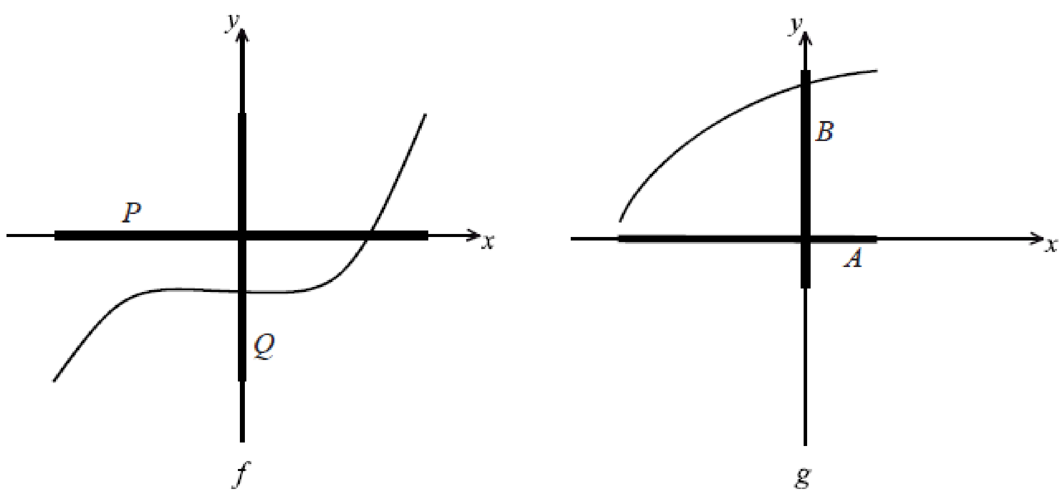
a. Below are the graphs of the two functions $F : P \to Q$ and $g : A \to B$. [4]



Determine, with reference to features of the graphs, whether the functions are injective and/or surjective.

b. Given two functions $h : X \to Y$ and $k : Y \to Z$. [9]

Show that

(i) if both h and k are injective then so is the composite function $k \circ h$;

(ii) if both h and k are surjective then so is the composite function $k \circ h$.

Consider the group $\{G, \times_{18}\}$ defined on the set $\{1, 5, 7, 11, 13, 17\}$ where $\times_{18}$ denotes multiplication modulo 18. The group $\{G, \times_{18}\}$ is shown in the following Cayley table.

$\times_{18}$	**1**	**5**	**7**	**11**	**13**	**17**
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

The subgroup of $\{G, \times_{18}\}$ of order two is denoted by $\{K, \times_{18}\}$.

a.i. Find the order of elements 5, 7 and 17 in $\{G, \times_{18}\}$. [4]

a.ii. State whether or not $\{G, \times_{18}\}$ is cyclic, justifying your answer. [2]

b. Write down the elements in set K. [1]

c. Find the left cosets of K in $\{G, \times_{18}\}$. [4]

A group $\{D, \times_3\}$ is defined so that $D = \{1, 2\}$ and $\times_3$ is multiplication modulo 3.

A function $f : \mathbb{Z} \to D$ is defined as $f : x \mapsto \begin{cases} 1, & x \text{ is even} \\ 2, & x \text{ is odd} \end{cases}$.

a. Prove that the function f is a homomorphism from the group $\{\mathbb{Z}, +\}$ to $\{D, \times_3\}$. [6]

b. Find the kernel of f. [3]

c. Prove that $\{\mathrm{Ker}(f), +\}$ is a subgroup of $\{\mathbb{Z}, +\}$. [4]

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a. Associativity and commutativity are two of the five conditions for a set S with the binary operation $*$ to be an Abelian group; state the other three conditions. [2]

b. The Cayley table for the binary operation $\odot$ defined on the set $T = \{p, q, r, s, t\}$ is given below. [15]

$\odot$	p	q	r	s	t
p	s	r	t	p	q
q	t	s	p	q	r
r	q	t	s	r	p
s	p	q	r	s	t
t	r	p	q	t	s

(i) Show that exactly three of the conditions for $\{T, \odot\}$ to be an Abelian group are satisfied, but that neither associativity nor commutativity are satisfied.
(ii) Find the proper subsets of T that are groups of order 2, and comment on your result in the context of Lagrange's theorem.
(iii) Find the solutions of the equation $(p \odot x) \odot x = x \odot p$.

The binary operation $*$ is defined by

$$a * b = a + b - 3 \text{ for } a,\ b \in \mathbb{Z}.$$

The binary operation $\circ$ is defined by

$$a \circ b = a + b + 3 \text{ for } a,\ b \in \mathbb{Z}.$$

Consider the group $\{\mathbb{Z},\ \circ\}$ and the bijection $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(a) = a - 6$.

a. Show that $\{\mathbb{Z},\ *\}$ is an Abelian group. [9]

b. Show that there is no element of order 2. [2]

c. Find a proper subgroup of $\{\mathbb{Z},\ *\}$. [2]

d. Show that the groups $\{\mathbb{Z},\ *\}$ and $\{\mathbb{Z},\ \circ\}$ are isomorphic. [3]

The set S is defined as the set of real numbers greater than 1.

The binary operation $*$ is defined on S by $x * y = (x - 1)(y - 1) + 1$ for all $x,\ y \in S$.

Let $a \in S$.

a. Show that $x * y \in S$ for all $x,\ y \in S$. [2]

b.i. Show that the operation $*$ on the set S is commutative. [2]

b.ii. Show that the operation $*$ on the set S is associative. [5]

c. Show that 2 is the identity element. [2]

d. Show that each element $a \in S$ has an inverse. [3]

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

a. Given that $R = (P \cap Q')'$, list the elements of R . [3]

b. For a set S , let S^* denote the set of all subsets of S , [5]

(i) find P^* ;
(ii) find $n(R^*)$.

The relation R is defined such that aRb if and only if $4^a - 4^b$ is divisible by 7, where $a,\ b \in \mathbb{Z}^+$.

The equivalence relation S is defined such that cSd if and only if $4^c - 4^d$ is divisible by 6, where $c,\ d \in \mathbb{Z}^+$.

a.i. Show that R is an equivalence relation. [6]

a.ii. Determine the equivalence classes of R. [3]

b. Determine the number of equivalence classes of S. [2]

An Abelian group, $\{G,\ *\}$, has 12 different elements which are of the form $a^i * b^j$ where $i \in \{1,\ 2,\ 3,\ 4\}$ and $j \in \{1,\ 2,\ 3\}$. The elements a and b satisfy $a^4 = e$ and $b^3 = e$ where e is the identity.

Let $\{H,\ *\}$ be the proper subgroup of $\{G,\ *\}$ having the maximum possible order.

a. State the possible orders of an element of $\{G,\ *\}$ and for each order give an example of an element of that order. [8]

b. (i) State a generator for $\{H,\ *\}$. [7]

(ii) Write down the elements of $\{H,\ *\}$.

(iii) Write down the elements of the coset of H containing a.

The relation R is defined such that xRy if and only if $|x| + |y| = |x + y|$ for $x,\ y,\ y \in \mathbb{R}$.

a.i. Show that R is reflexive. [2]

a.ii. Show that R is symmetric. [2]

b. Show, by means of an example, that R is not transitive. [4]

The group G has a unique element, h , of order 2.

(i) Show that ghg^{-1} has order 2 for all $g \in G$.

(ii) Deduce that $gh = hg$ for all $g \in G$.

Two functions, F and G , are defined on $A = \mathbb{R} \backslash \{0,\ 1\}$ by

$$F(x) = \frac{1}{x},\ G(x) = 1 - x,\ \text{for all } x \in A.$$

(a) Show that under the operation of composition of functions each function is its own inverse.

(b) F and G together with four other functions form a closed set under the operation of composition of functions.

Find these four functions.

The binary operation $*$ is defined for $x,\ y \in S = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ by

$$x * y = (x^3y - xy) \text{ mod } 7.$$

a. Find the element e such that $e * y = y$, for all $y \in S$. [2]

b. (i) Find the least solution of $x * x = e$. [5]

(ii) Deduce that $(S,\ *)$ is not a group.

c. Determine whether or not e is an identity element. [3]

All of the relations in this question are defined on $\mathbb{Z} \backslash \{0\}$.

a. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow x + y > 7$ is [4]

(i) reflexive;

(ii) symmetric;

(iii) transitive.

b. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow -2 < x - y < 2$ is [4]

(i) reflexive;

(ii) symmetric;

(iii) transitive.

c. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow xy > 0$ is [4]

(i) reflexive;

(ii) symmetric;

(iii) transitive.

d. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow \frac{x}{y} \in \mathbb{Z}$ is [4]

(i) reflexive;
(ii) symmetric;
(iii) transitive.

e. One of the relations from parts (a), (b), (c) and (d) is an equivalence relation. [3]

For this relation, state what the equivalence classes are.

Let $A = \{a,\ b\}$.

Let the set of all these subsets be denoted by $P(A)$. The binary operation symmetric difference, Δ , is defined on $P(A)$ by

$X\Delta Y = (X\backslash Y) \cup (Y\backslash X)$ where X , $Y \in P(A)$.

Let $\mathbb{Z}_4 = \{0,\ 1,\ 2,\ 3\}$ and $+_4$ denote addition modulo 4.

Let S be any non-empty set. Let $P(S)$ be the set of all subsets of S . For the following parts, you are allowed to assume that $\Delta, \cup$ and $\cap$ are associative.

a. Write down all four subsets of A . [1]

b. Construct the Cayley table for $P(A)$ under Δ . [3]

c. Prove that $\{P(A),\ \Delta\}$ is a group. You are allowed to assume that Δ is associative. [3]

d. Is $\{P(A), \Delta\}$ isomorphic to $\{\mathbb{Z}_4,\ +_4\}$? Justify your answer. [2]

e. (i) State the identity element for $\{P(S), \Delta\}$. [4]

(ii) Write down X^{-1} for $X \in P(S)$.

(iii) Hence prove that $\{P(S), \Delta\}$ is a group.

f. Explain why $\{P(S), \cup\}$ is not a group. [1]

g. Explain why $\{P(S), \cap\}$ is not a group. [1]

The binary operation $*$ is defined on the set $T = \{0,\ 2,\ 3,\ 4,\ 5,\ 6\}$ by $a * b = (a + b - ab)(\bmod 7),\ a,\ b \in T$.

a. Copy and complete the following Cayley table for $\{T,\ *\}$. [4]

$*$	0	2	3	4	5	6
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3	6				
4	4	5				
5	5	4				
6	6	3				

b. Prove that $\{T,\ *\}$ forms an Abelian group. [7]

c. Find the order of each element in T. [4]

d. Given that $\{H,\ *\}$ is the subgroup of $\{T,\ *\}$ of order 2, partition T into the left cosets with respect to H. [3]

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $f(x,\ y) = (2x^3 + y^3,\ x^3 + 2y^3)$.

a. Show that f is a bijection. [12]

b. Hence write down the inverse function $f^{-1}(x,\ y)$. [1]

Let A be the set $\{x | x \in \mathbb{R},\ x \neq 0\}$. Let B be the set $\{x | x \in]-1,\ +1[,\ x \neq 0\}$.

A function $f : A \to B$ is defined by $f(x) = \frac{2}{\pi}\arctan(x)$.

Let D be the set $\{x | x \in \mathbb{R},\ x > 0\}$.

A function $g : \mathbb{R} \to D$ is defined by $g(x) = \mathrm{e}^x$.

a. (i) Sketch the graph of $y = f(x)$ and hence justify whether or not f is a bijection. [13]

(ii) Show that A is a group under the binary operation of multiplication.

(iii) Give a reason why B is not a group under the binary operation of multiplication.

(iv) Find an example to show that $f(a \times b) = f(a) \times f(b)$ is not satisfied for all $a,\ b \in A$.

b. (i) Sketch the graph of $y = g(x)$ and hence justify whether or not g is a bijection. [8]

(ii) Show that $g(a + b) = g(a) \times g(b)$ for all $a,\ b \in \mathbb{R}$.

(iii) Given that $\{\mathbb{R},\ +\}$ and $\{D,\ \times\}$ are both groups, explain whether or not they are isomorphic.

(a) Show that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x,\ y) = (2x + y,\ x - y)$ is a bijection.

(b) Find the inverse of f.

The binary operation $*$ is defined on $\mathbb{R}$ as follows. For any elements a , $b \in \mathbb{R}$

$$a * b = a + b + 1.$$

a. (i) Show that $*$ is commutative. [5]

(ii) Find the identity element.

(iii) Find the inverse of the element a .

b. The binary operation $\cdot$ is defined on $\mathbb{R}$ as follows. For any elements a , $b \in \mathbb{R}$ [6]

$a \cdot b = 3ab$. The set S is the set of all ordered pairs $(x,\ y)$ of real numbers and the binary operation $\odot$ is defined on the set S as $(x_1,\ y_1) \odot (x_2,\ y_2) = (x_1 * x_2,\ y_1 \cdot y_2)$.
Determine whether or not $\odot$ is associative.

(a) Draw the Cayley table for the set of integers $G = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6, $+_6$.

(b) Show that $\{G,\ +_6\}$ is a group.

(c) Find the order of each element.

(d) Show that $\{G,\ +_6\}$ is cyclic and state its generators.

(e) Find a subgroup with three elements.

(f) Find the other proper subgroups of $\{G,\ +_6\}$.

The function $f : [0,\ \infty[\to [0,\ \infty[$ is defined by $f(x) = 2\mathrm{e}^x + \mathrm{e}^{-x} - 3$.

(a) Find $f'(x)$.

(b) Show that f is a bijection.

(c) Find an expression for $f^{-1}(x)$.

The universal set contains all the positive integers less than 30. The set A contains all prime numbers less than 30 and the set B contains all positive integers of the form $3 + 5n$ $(n \in \mathbb{N})$ that are less than 30. Determine the elements of

a. $A \setminus B$; [4]

b. $A \Delta B$. [3]

The binary operation $*$ is defined on $\mathbb{N}$ by $a * b = 1 + ab$.

Determine whether or not $*$

a. is closed; [2]

b. is commutative; [2]

c. is associative; [3]

d. has an identity element. [3]

A group with the binary operation of multiplication modulo 15 is shown in the following Cayley table.

$\times_{15}$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	a	b	c
13	13	11	7	1	14	d	e	f
14	14	13	11	8	7	g	h	i

a. Find the values represented by each of the letters in the table. [3]

b. Find the order of each of the elements of the group. [3]

c. Write down the three sets that form subgroups of order 2. [2]

d. Find the three sets that form subgroups of order 4. [4]

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a. Given that p , q and r are elements of a group, prove the left-cancellation rule, *i.e.* $pq = pr \Rightarrow q = r$. [4]

Your solution should indicate which group axiom is used at each stage of the proof.

b. Consider the group G , of order 4, which has distinct elements a , b and c and the identity element e . [10]

(i) Giving a reason in each case, explain why ab cannot equal a or b .

(ii) Given that c is self inverse, determine the two possible Cayley tables for G .

(iii) Determine which one of the groups defined by your two Cayley tables is isomorphic to the group defined by the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ under multiplication of complex numbers. Your solution should include a correspondence between a, b, c, e and $1, -1, \mathrm{i}, -\mathrm{i}$.

A binary operation is defined on $\{-1, 0, 1\}$ by

$$A \odot B = \begin{cases} -1, & \text{if } |A| < |B| \\ 0, & \text{if } |A| = |B| \\ 1, & \text{if } |A| > |B| . \end{cases}$$

(a) Construct the Cayley table for this operation.

(b) Giving reasons, determine whether the operation is

(i) closed;

(ii) commutative;

(iii) associative.

Sets X and Y are defined by $X =]0,\ 1[;\ Y = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$.

a. (i) Sketch the set $X \times Y$ in the Cartesian plane. [5]

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$.

b. Consider the function $f : X \times Y \to \mathbb{R}$ defined by $f(x,\ y) = x + y$ and the function $g : X \times Y \to \mathbb{R}$ defined by $g(x,\ y) = xy$. [10]

(i) Find the range of the function f.

(ii) Find the range of the function g.

(iii) Show that f is an injection.

(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x,\ y) = \frac{1}{2}$.

Let $f : G \to H$ be a homomorphism of finite groups.

a. Prove that $f(e_G) = e_H$, where e_G is the identity element in G and e_H is the identity element in H. [2]

b. (i) Prove that the kernel of $f,\ K = \mathrm{Ker}(f)$, is closed under the group operation. [6]

(ii) Deduce that K is a subgroup of G.

c. (i) Prove that $gkg^{-1} \in K$ for all $g \in G,\ k \in K$. [6]

(ii) Deduce that each left coset of K in G is also a right coset.

Let X and Y be sets. The functions $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f$ is the identity function on X.

a. Prove that: [6]

(i) f is an injection,

(ii) g is a surjection.

b. Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions f and g to show that g is not necessarily a bijection. [3]

Let $(H, \ *)$ be a subgroup of the group $(G, \ *)$.

Consider the relation R defined in G by xRy if and only if $y^{-1} * x \in H$.

(a) Show that R is an equivalence relation on G.

(b) Determine the equivalence class containing the identity element.

Consider the set A consisting of all the permutations of the integers $1, 2, 3, 4, 5$.

a. Two members of A are given by $p = (1\ 2\ 5)$ and $q = (1\ 3)(2\ 5)$. [4]

Find the single permutation which is equivalent to $q \circ p$.

b. State a permutation belonging to A of order [3]

(i) 4;

(ii) 6.

c. Let $P =$ {all permutations in A where exactly two integers change position}, [4]

and $Q =$ {all permutations in A where the integer 1 changes position}.

(i) List all the elements in $P \cap Q$.

(ii) Find $n(P \cap Q')$.

Given the sets A and B, use the properties of sets to prove that $A \cup (B' \cup A)' = A \cup B$, justifying each step of the proof.

(a) Write down why the table below is a Latin square.

$$\begin{array}{c|ccccc} & d & e & b & a & c \\ \hline d & c & d & e & b & a \\ e & d & e & b & a & c \\ b & a & b & d & c & e \\ a & b & a & c & e & d \\ c & e & c & a & d & b \end{array}$$

(b) Use Lagrange's theorem to show that the table is not a group table.

Let $p = 2^k + 1,\ k \in \mathbb{Z}^+$ be a prime number and let G be the group of integers 1, 2, ..., $p - 1$ under multiplication defined modulo p.

By first considering the elements $2^1,\ 2^2,\ ...,\ 2^k$ and then the elements $2^{k+1},\ 2^{k+2},\ \ldots$, show that the order of the element 2 is $2k$.

Deduce that $k = 2^n$ for $n \in \mathbb{N}$.

Prove that $(A \cap B)\backslash(A \cap C) = A \cap (B\backslash C)$ where A, B and C are three subsets of the universal set U.

Let $\{G,\ *\}$ be a finite group and let H be a non-empty subset of G . Prove that $\{H,\ *\}$ is a group if H is closed under $*$.

The group $\{G,\ *\ \}$ has identity e_G and the group $\{H,\ \circ\}$ has identity e_H. A homomorphism f is such that $f : G \to H$. It is given that $f(e_G) = e_H$.

a. Prove that for all $a \in G,\ f(a^{-1}) = (f(a))^{-1}$. [4]

b. Let $\{H,\ \circ\}$ be the cyclic group of order seven, and let p be a generator. [4]

   Let $x \in G$ such that $f(x) = p^2$.

   Find $f(x^{-1})$.

c. Given that $f(x * y) = p$, find $f(y)$. [4]

H and K are subgroups of a group G. By considering the four group axioms, prove that $H \cap K$ is also a subgroup of G.

Prove that set difference is not associative.

Define $f : \mathbb{R}\backslash\{0.5\} \to \mathbb{R}$ by $f(x) = \frac{4x+1}{2x-1}$.

a. Prove that f is an injection. [4]

b. Prove that f is not a surjection. [4]

Consider the sets

$$G = \left\{\frac{n}{6^i}|n \in \mathbb{Z},\ i \in \mathbb{N}\right\},\ H = \left\{\frac{m}{3^j}|m \in \mathbb{Z},\ j \in \mathbb{N}\right\}.$$

a. Show that $(G,\ +)$ forms a group where $+$ denotes addition on $\mathbb{Q}$. Associativity may be assumed. [5]

b. Assuming that $(H,\ +)$ forms a group, show that it is a proper subgroup of $(G,\ +)$. [4]

c. The mapping $\phi : G \to G$ is given by $\phi(g) = g + g$, for $g \in G$. [7]

Prove that ϕ is an isomorphism.

Consider the following functions

$f :]1,\ +\infty[\to \mathbb{R}^+$ where $f(x) = (x-1)(x+2)$

$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $g(x,\ y) = (\sin(x+y),\ x+y)$

$h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $h(x,\ y) = (x+3y,\ 2x+y)$

(a) Show that f is bijective.

(b) Determine, with reasons, whether

(i) g is injective;

(ii) g is surjective.

(c) Find an expression for $h^{-1}(x,\ y)$ and hence justify that h has an inverse function.

---

a. Let $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ f(m,\ x) = (-1)^m x$. Determine whether f is [4]

(i) surjective;

(ii) injective.

b. P is the set of all polynomials such that $P = \left\{\sum\limits_{i=0}^{n} a_i x^i | n \in \mathbb{N}\right\}$. [4]

Let $g : P \to P,\ g(p) = xp$. Determine whether g is

(i) surjective;

(ii) injective.

c. Let $h : \mathbb{Z} \to \mathbb{Z}^+$, $h(x) = \left\{\begin{array}{cc} 2x, & x > 0 \\ 1-2x, & x \leqslant 0 \end{array}\right\}$. Determine whether h is [7]

(i) surjective;

(ii) injective.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f : x \to \begin{cases} 1, \ x \geq 0 \\ -1, \ x < 0 \end{cases}$.

a. Prove that f is [4]

(i) not injective;

(ii) not surjective.

b. The relation R is defined for $a, \ b \in \mathbb{R}$ so that aRb if and only if $f(a) \times f(b) = 1$. [8]

Show that R is an equivalence relation.

c. The relation R is defined for $a, \ b \in \mathbb{R}$ so that aRb if and only if $f(a) \times f(b) = 1$. [2]

State the equivalence classes of R.

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x, \ y) = \left(xy^2, \frac{x}{y}\right)$.

Show that f is a bijection.

Let G be a finite cyclic group.

(a) Prove that G is Abelian.

(b) Given that a is a generator of G, show that a^{-1} is also a generator.

(c) Show that if the order of G is five, then all elements of G, apart from the identity, are generators of G.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 2\mathrm{e}^{x} - \mathrm{e}^{-x}.$$

(a) Show that f is a bijection.

(b) Find an expression for $f^{-1}(x)$.

The set of all permutations of the list of the integers 1, 2, 3 4 is a group, S_4, under the operation of function composition.

In the group S_4 let $p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$.

a. Determine the order of S_4. [2]

b. Find the proper subgroup H of order 6 containing p_1, p_2 and their compositions. Express each element of H in cycle form. [5]

c. Let $f\colon S_4 \to S_4$ be defined by $f(p) = p \circ p$ for $p \in S_4$. [5]

Using p_1 and p_2, explain why f is not a homomorphism.

---

a. The relation aRb is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if and only if ab is the square of a positive integer. [10]

(i) Show that R is an equivalence relation.

(ii) Find the equivalence classes of R that contain more than one element.

b. Given the group $(G, *)$, a subgroup $(H, *)$ and $a, b \in G$, we define $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation. [9]

Set $S = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ and a binary operation $\circ$ on S is defined as $x_i \circ x_j = x_k$, where $i + j \equiv k(\bmod 6)$.

(a) (i) Construct the Cayley table for $\{S, \circ\}$ and hence show that it is a group.

(ii) Show that $\{S, \circ\}$ is cyclic.

(b) Let $\{G, *\}$ be an Abelian group of order 6. The element $a \in$ G has order 2 and the element $b \in$ G has order 3.

(i) Write down the six elements of $\{G, *\}$.

(ii) Find the order of a $*\, b$ and hence show that $\{G, *\}$ is isomorphic to $\{S, \circ\}$.

The function *f* is defined by

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}},\ x \in \mathbb{R}.$$

(a) Find the range of *f*.

(b) Prove that *f* is an injection.

(c) Taking the codomain of *f* to be equal to the range of *f*, find an expression for $f^{-1}(x)$.

The relation R is defined on $\mathbb{Z} \times \mathbb{Z}$ such that $(a, b)R(c, d)$ if and only if $a - c$ is divisible by 3 and $b - d$ is divisible by 2.

(a) Prove that R is an equivalence relation.

(b) Find the equivalence class for (2, 1).

(c) Write down the five remaining equivalence classes.

The binary operation $*$ is defined on the set $S = \{0, 1, 2, 3\}$ by

$$a * b = a + 2b + ab(\bmod 4).$$

(a) (i) Construct the Cayley table.

(ii) Write down, with a reason, whether or not your table is a Latin square.

(b) (i) Write down, with a reason, whether or not $*$ is commutative.

(ii) Determine whether or not $*$ is associative, justifying your answer.

(c) Find all solutions to the equation $x * 1 = 2 * x$, for $x \in S$.

(a) Find the six roots of the equation $z^6 - 1 = 0$, giving your answers in the form $r \operatorname{cis} \theta$, $r \in \mathbb{R}^+$, $0 \leqslant \theta < 2\pi$.

(b) (i) Show that these six roots form a group G under multiplication of complex numbers.

(ii) Show that G is cyclic and find all the generators.

(iii) Give an example of another group that is isomorphic to G, stating clearly the corresponding elements in the two groups.

---

a. The relation R is defined on $\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied. [5]

b. The relation S is defined on $\mathbb{Z}^+$ by aSb if and only if $a^2 \equiv b^2(\bmod 6)$. [9]

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members.

The groups $\{K, *\}$ and $\{H, \odot\}$ are defined by the following Cayley tables.

G

$*$	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	A	E
C	C	B	E	A

H

$\odot$	e	a
e	e	a
a	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

Three functions mapping $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ are defined by

$$f_1(m,\ n) = m - n + 4;\ \ f_2(m,\ n) = |m|\ ;\ \ f_3(m,\ n) = m^2 - n^2.$$

Two functions mapping $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ are defined by

$$g_1(k) = (2k,\ k);\ \ g_2(k) = (k,\ \ |k|)\,.$$

(a) Find the range of

(i) $f_1 \circ g_1$;

(ii) $f_3 \circ g_2$.

(b) Find all the solutions of $f_1 \circ g_2(k) = f_2 \circ g_1(k)$.

(c) Find all the solutions of $f_3(m,\ n) = p$ in each of the cases $p = 1$ and $p = 2$.

$\{G,\ *\}$ is a group with identity element e. Let $a,\ b \in G$.

a. State Lagrange's theorem. [2]

b. Verify that the inverse of $a * b^{-1}$ is equal to $b * a^{-1}$. [3]

c. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$. Let R be a relation defined on G by [8]

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

Prove that R is an equivalence relation, indicating clearly whenever you are using one of the four properties required of a group.

d. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$ .Let R be a relation defined on G by [3]

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

Show that $aRb \Leftrightarrow a \in Hb$, where Hb is the right coset of H containing b.

e. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$ .Let R be a relation defined on G by [3]

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

It is given that the number of elements in any right coset of H is equal to the order of H.

Explain how this fact together with parts (c) and (d) prove Lagrange's theorem.

(a) Given a set U, and two of its subsets A and B, prove that

$$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B),\ \text{where}\ A \backslash B = A \cap B'.$$

(b) Let $S = \{A,\ B,\ C,\ D\}$ where $A = \emptyset,\ B = \{0\},\ C = \{0,\ 1\}$ and $D = \{0, 1, 2\}$.
State, with reasons, whether or not each of the following statements is true.

(i) The operation \ is closed in S.

(ii) The operation $\cap$ has an identity element in S but not all elements have an inverse.

(iii) Given $Y \in S$, the equation $X \cup Y = Y$ always has a unique solution for X in S.

The relation R is defined on $\mathbb{Z}$ by xRy if and only if $x^2y \equiv y \bmod 6$.

a. Show that the product of three consecutive integers is divisible by 6. [2]

b. Hence prove that R is reflexive. [3]

c. Find the set of all y for which $5Ry$. [3]

d. Find the set of all y for which $3Ry$. [2]

e. Using your answers for (c) and (d) show that R is not symmetric. [2]

Determine, giving reasons, which of the following sets form groups under the operations given below. Where appropriate you may assume that multiplication is associative.

(a) $\mathbb{Z}$ under subtraction.

(b) The set of complex numbers of modulus 1 under multiplication.

(c) The set $\{1, 2, 4, 6, 8\}$ under multiplication modulo 10.

(d) The set of rational numbers of the form

$$\frac{3m+1}{3n+1}, \text{ where } m,\ n \in \mathbb{Z}$$

under multiplication.

Consider the set S defined by $S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.

You may assume that $+$ (addition) and $\times$ (multiplication) are associative binary operations on $\mathbb{Q}$.

a. (i) Write down the six smallest non-negative elements of S. [9]

(ii) Show that $\{S,\ +\}$ is a group.

(iii) Give a reason why $\{S,\ \times\}$ is not a group. Justify your answer.

b. The relation R is defined on S by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$. [10]

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence classes.

The binary operation Δ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

Δ	1	2	3	4	5
1	1	1	2	3	4
2	1	2	1	2	3
3	2	1	3	1	2
4	3	2	1	4	1
5	4	3	2	1	5

(a) State whether S is closed under the operation Δ and justify your answer.

(b) State whether Δ is commutative and justify your answer.

(c) State whether there is an identity element and justify your answer.

(d) Determine whether Δ is associative and justify your answer.

(e) Find the solutions of the equation $a\Delta b = 4\Delta b$, for $a \neq 4$.

The binary operation $*$ is defined for $a, b \in \mathbb{Z}^+$ by

$$a * b = a + b - 2.$$

(a) Determine whether or not $*$ is

(i) closed,

(ii) commutative,

(iii) associative.

(b) (i) Find the identity element.

(ii) Find the set of positive integers having an inverse under $*$.

A, B, C and D are subsets of $\mathbb{Z}$.

$A = \{m| \, m \text{ is a prime number less than } 15\}$

$B = \{m| \, m^4 = 8m\}$

$C = \{m| \, (m+1)(m-2) < 0\}$

$D = \{m| \, m^2 < 2m + 4\}$

(a) List the elements of each of these sets.

(b) Determine, giving reasons, which of the following statements are true and which are false.

(i) $n(D) = n(B) + n(B \cup C)$

(ii) $D \backslash B \subset A$

(iii) $B \cap A' = \emptyset$

(iv) $n(B \Delta C) = 2$

(a) Consider the set $A = \{1, 3, 5, 7\}$ under the binary operation $*$, where $*$ denotes multiplication modulo 8.

(i) Write down the Cayley table for $\{A, *\}$.

(ii) Show that $\{A, *\}$ is a group.

(iii) Find all solutions to the equation $3 * x * 7 = y$. Give your answers in the form (x, y).

(b) Now consider the set $B = \{1, 3, 5, 7, 9\}$ under the binary operation $\otimes$, where $\otimes$ denotes multiplication modulo 10. Show that $\{B, \otimes\}$ is not a group.

(c) Another set C can be formed by removing an element from B so that $\{C, \otimes\}$ is a group.

(i) State which element has to be removed.

(ii) Determine whether or not $\{A, *\}$ and $\{C, \otimes\}$ are isomorphic.

Let $\{G, *\}$ be a finite group of order n and let H be a non-empty subset of G.

(a) Show that any element $h \in H$ has order smaller than or equal to n.

(b) If H is closed under $*$, show that $\{H, *\}$ is a subgroup of $\{G, *\}$.

The group $\{G, *\}$ is Abelian and the bijection $f: G \to G$ is defined by $f(x) = x^{-1}$, $x \in G$.

Show that f is an isomorphism.

The group G has a subgroup H. The relation R is defined on G by xRy if and only if $xy^{-1} \in H$, for $x, y \in G$.

a. Show that R is an equivalence relation. [8]

b. The Cayley table for G is shown below. [6]

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

The subgroup H is given as $H = \{e, a^2b\}$.

(i) Find the equivalence class with respect to R which contains ab.

(ii) Another equivalence relation ρ is defined on G by $x\rho y$ if and only if $x^{-1}y \in H$, for $x, y \in G$. Find the equivalence class with respect to ρ which contains ab.

The relation R is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by aRb if and only if $a(a+1) \equiv b(b+1)(\bmod 5)$.

a. Show that R is an equivalence relation. [6]

b. Show that the equivalence defining R can be written in the form [3]

$$(a-b)(a+b+1) \equiv 0(\bmod 5).$$

c. Hence, or otherwise, determine the equivalence classes. [4]

Consider the set $S_3 = \{p, q, r, s, t, u\}$ of permutations of the elements of the set $\{1, 2, 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $\circ$ denote composition of permutations, so $a \circ b$ means b followed by a. You may assume that $(S_3, \circ)$ forms a group.

a. Complete the following Cayley table [4]

$\circ$	p	q	r	s	t	u
p						
q			t			s
r		u		t	s	q
s		t	u			r
t		s	q	r		
u		r	s	q		

[5 marks]

b. (i) State the inverse of each element. [6]

(ii) Determine the order of each element.

c. Write down the subgroups containing [2]

(i) r,

(ii) u.

The permutation p_1 of the set $\{1, 2, 3, 4\}$ is defined by

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

(a) (i) State the inverse of p_1.

(ii) Find the order of p_1.

(b) Another permutation p_2 is defined by

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

(i) Determine whether or not the composition of p_1 and p_2 is commutative.

(ii) Find the permutation p_3 which satisfies

$$p_1 p_3 p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Let R be a relation on the set $\mathbb{Z}$ such that $aRb \Leftrightarrow ab \geqslant 0$, for $a, b \in \mathbb{Z}$.

(a) Determine whether R is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

(b) Write down with a reason whether or not R is an equivalence relation.

The relation R is defined for a , $b \in \mathbb{Z}^+$ such that aRb if and only if $a^2 - b^2$ is divisible by 5.

a. Show that R is an equivalence relation. [6]

b. Identify the three equivalence classes. [4]

Let G be a group of order 12 with identity element e.

Let $a \in G$ such that $a^6 \neq e$ and $a^4 \neq e$.

a. (i) Prove that G is cyclic and state two of its generators. [9]

(ii) Let H be the subgroup generated by a^4. Construct a Cayley table for H.

b. State, with a reason, whether or not it is necessary that a group is cyclic given that all its proper subgroups are cyclic. [2]

(a) Show that $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ forms a group of complex numbers G under multiplication.

(b) Consider $S = \{e,\ a,\ b,\ a * b\}$ under an associative operation $*$ where e is the identity element. If $a * a = b * b = e$ and $a * b = b * a$, show that

(i) $a * b * a = b$,

(ii) $a * b * a * b = e$.

(c) (i) Write down the Cayley table for $H = \{S\ ,*\}$.

(ii) Show that H is a group.

(iii) Show that H is an Abelian group.

(d) For the above groups, G and H, show that one is cyclic and write down why the other is not. Write down all the generators of the cyclic group.

(e) Give a reason why G and H are not isomorphic.

The relations R and S are defined on quadratic polynomials P of the form

$$P(z) = z^2 + az + b\text{, where } a\ , b \in \mathbb{R}\ , z \in \mathbb{C}\ .$$

(a) The relation R is defined by $P_1 R P_2$ if and only if the sum of the two zeros of P_1 is equal to the sum of the two zeros of P_2.

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence class containing $z^2 - 4z + 5$.

(b) The relation S is defined by $P_1 S P_2$ if and only if P_1 and P_2 have at least one zero in common. Determine whether or not S is transitive.

The relation R is defined on ordered pairs by

$$(a,\ b)R(c,\ d) \text{ if and only if } ad = bc \text{ where } a,\ b,\ c,\ d \in \mathbb{R}^+.$$

(a) Show that R is an equivalence relation.

(b) Describe, geometrically, the equivalence classes.

Consider the set $S = \{1, 3, 5, 7, 9, 11, 13\}$ under the binary operation multiplication modulo 14 denoted by $\times_{14}$.

a. Copy and complete the following Cayley table for this binary operation. [4]

$\times_{14}$	1	3	5	7	9	11	13
1	1	3	5	7	9	11	13
3	3				13	5	11
5	5				3	13	9
7	7						
9	9	13	3				
11	11	5	13				
13	13	11	9				

b. Give one reason why $\{S, \times_{14}\}$ is not a group. [1]

c. Show that a new set G can be formed by removing one of the elements of S such that $\{G, \times_{14}\}$ is a group. [5]

d. Determine the order of each element of $\{G, \times_{14}\}$. [4]

e. Find the proper subgroups of $\{G, \times_{14}\}$. [2]

The binary operator multiplication modulo 14, denoted by $*$, is defined on the set $S = \{2, 4, 6, 8, 10, 12\}$.

a. Copy and complete the following operation table. [4]

$*$	2	4	6	8	10	12
2						
4	8	2	10	4	12	6
6						
8						
10	6	12	4	10	2	8
12						

b. (i) Show that $\{S, *\}$ is a group. [11]

(ii) Find the order of each element of $\{S, *\}$.

(iii) Hence show that $\{S, *\}$ is cyclic and find all the generators.

c. The set T is defined by $\{x * x : x \in S\}$. Show that $\{T, *\}$ is a subgroup of $\{S, *\}$. [3]

The group $\{G, *\}$ is defined on the set G with binary operation $*$. H is a subset of G defined by $H = \{x : x \in G, a * x * a^{-1} = x \text{ for all } a \in G\}$. Prove that $\{H, *\}$ is a subgroup of $\{G, *\}$.

The following Cayley table for the binary operation multiplication modulo 9, denoted by $*$, is defined on the set $S = \{1, 2, 4, 5, 7, 8\}$.

*	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8				
5	5	1				
7	7	5				
8	8	7				

a. Copy and complete the table. [3]

b. Show that $\{S, *\}$ is an Abelian group. [5]

c. Determine the orders of all the elements of $\{S, *\}$. [3]

d. (i) Find the two proper subgroups of $\{S, *\}$. [4]

(ii) Find the coset of each of these subgroups with respect to the element 5.

e. Solve the equation $2 * x * 4 * x * 4 = 2$. [4]

The binary operation multiplication modulo 10, denoted by $\times_{10}$, is defined on the set T = {2 , 4 , 6 , 8} and represented in the following Cayley table.

$\times_{10}$	**2**	**4**	**6**	**8**
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

a. Show that $\{T, \times_{10}\}$ is a group. (You may assume associativity.) [4]

b. By making reference to the Cayley table, explain why T is Abelian. [1]

c.i. Find the order of each element of $\{T, \times_{10}\}$. [3]

c.ii. Hence show that $\{T, \times_{10}\}$ is cyclic and write down all its generators. [3]

d. The binary operation multiplication modulo 10, denoted by $\times_{10}$, is defined on the set V = {1, 3 ,5 ,7 ,9}. [2]

Show that $\{V, \times_{10}\}$ is not a group.

Consider the sets A = {1, 3, 5, 7, 9} , B = {2, 3, 5, 7, 11} and C = {1, 3, 7, 15, 31} .

a.i. Find $(A \cup B) \cap (A \cup C)$. [3]

a.ii. Verify that $A \setminus C \neq C \setminus A$. [2]

b. Let S be a set containing n elements where $n \in \mathbb{N}$. [3]

Show that S has 2^n subsets.

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a. Consider the following Cayley table for the set $G = \{1, 3, 5, 7, 9, 11, 13, 15\}$ under the operation $\times_{16}$, where $\times_{16}$ denotes multiplication modulo 16. [7]

$\times_{16}$	1	3	5	7	9	11	13	15
1	1	3	5	7	9	11	13	15
3	3	a	15	5	11	b	7	c
5	5	15	9	3	13	7	1	11
7	7	d	3	1	e	13	f	9
9	9	11	13	g	1	3	5	7
11	11	h	7	13	3	9	i	5
13	13	7	1	11	5	j	9	3
15	15	13	11	9	7	5	3	1

(i) Find the values of a, b, c, d, e, f, g, h, i and j.

(ii) Given that $\times_{16}$ is associative, show that the set G, together with the operation $\times_{16}$, forms a group.

b. The Cayley table for the set $H = \{e, a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$ under the operation $*$, is shown below. [8]

$*$	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
e	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	e	b_4	b_3	b_1	b_2
a_2	a_2	a_3	e	a_1	b_2	b_1	b_4	b_3
a_3	a_3	e	a_1	a_2	b_3	b_4	b_2	b_1
b_1	b_1	b_3	b_2	b_4	e	a_2	a_1	a_3
b_2	b_2	b_4	b_1	b_3	a_2	e	a_3	a_1
b_3	b_3	b_2	b_4	b_1	a_3	a_1	e	a_2
b_4	b_4	b_1	b_3	b_2	a_1	a_3	a_2	e

(i) Given that $*$ is associative, show that H together with the operation $*$ forms a group.

(ii) Find two subgroups of order 4.

c. Show that $\{G, \times_{16}\}$ and $\{H, *\}$ are not isomorphic. [2]

d. Show that $\{H, *\}$ is not cyclic. [3]

a. The function $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(n) = |n| - 1$ for $n \in \mathbb{Z}$. Show that g is neither surjective nor injective. [2]

b. The set S is finite. If the function $f : S \to S$ is injective, show that f is surjective. [2]

c. Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. [3]

Consider the functions $f : A \to B$ and $g : B \to C$.

a. Show that if both f and g are injective, then $g \circ f$ is also injective. [3]

b. Show that if both f and g are surjective, then $g \circ f$ is also surjective. [4]

c. Show, using a single counter example, that both of the converses to the results in part (a) and part (b) are false. [3]

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2x + 1 & \text{for } x \leqslant 2 \\ x^2 - 2x + 5 & \text{for } x > 2. \end{cases}$$

a. (i) Sketch the graph of f. [5]

(ii) By referring to your graph, show that f is a bijection.

b. Find $f^{-1}(x)$. [8]

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a. Determine, using Venn diagrams, whether the following statements are true. [6]

(i) $A' \cup B' = (A \cup B)'$

(ii) $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

b. Prove, without using a Venn diagram, that $A \backslash B$ and $B \backslash A$ are disjoint sets. [4]
